Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021-8928(02)00104-1

J. Appl. Maths Mechs, Vol. 66, No. 5, pp. 725–737, 2002

0021-8928/02/$—see front matter

AN EXAMPLE OF THE CONSTRUCTION OF A MULTIVALUED QUASI-STRATEGY BY ITERATIVE METHODS†

K. V. KORLYAKOV and A. G. CHENTSOV

Ekaterinburg

e-mail: chentsov@imm.uran.ru

(*Received* 18 *October* 2001)

An example is considered of the application of a new modification of the well-known programmed iteration method (PIM) for solving a simple problem of control under conditions of noise. It is shown that the construction of a control procedure – a multivalued quasi-strategy – which solves the initial problem by the "direct" version of the PIM, working in the space of multivalued mappings, requires the performance of an infinite number of iterations. The structure of thee iterations is established on the basis of the duality of "direct" and "indirect" versions of the PIM (the latter having been used previously to construct value functions and stable bridges in Krasovskii's sense). © 2003 Elsevier Science Ltd. All rights reserved.

The structure of the solution of many differential games (DGs) [1–5] is determined by the basic Alternative Theorem of Krasovskii and Subbotin (see [3]), which states that the space of positions in a pursuit–evasion game admits of an alternative partition into the sum of two sets, one of which – the positional absorption set (a stable bridge) – is defined as the set of all initial positions from which the pursuer can guarantee that the trajectories of the system will reach the target set. The construction of stable bridges used programmed constructions (see, e.g. [3, 4, 6, 7]) which, under certain regularity conditions [2–4, 7], guarantee direct passage from programmed control to synthesis. If those conditions are not satisfied, this approach becomes more complicated and other methods are needed, one of which is the programmed iteration method (PIM) [8–15] (see also [5, Chaps IV and V]). It has turned out that, for certain non-regular DGs, a solution (the value of the game, a stable bridge) may be constructed by means of just two iterated programmed absorptions [8, 9], though in other cases the whole infinite sequence of programmed iterations must be implemented (see [14, 15]). Examples have been described (see [14] and various later publications) in which the solution defined by the PIM is obtained in any prescribed number of iterations.

The above examples of DGs are "indirect" in the sense of the construction of the corresponding control procedures (the PIM is used to generate a "go-between" in the form of a value function or stable bridge). These procedures may be defined on the basis of feedback (Krasovskii's constructions of extremal aiming and extremal shift) or as quasi-strategies (see [16, 17] for single-valued quasi-strategies and [5, 8–11, 13, 14] for multivalued quasi-strategies). Later, a "direct" version of the PIM, which works in the space of multivalued mappings, was constructed [18–22]; in problems of differential game theory it permitted direct iterative construction of a multivalued quasi-strategy, like those described previously ([5], Chap. IV and [8–11, 13, 14]), which solves a corresponding control problem.

In many simple DGs, the "direct" version of the PIM generates a solution in two iterations (see [18, 22]). Among these examples are regular DGs, in which a value function and a stable bridge are found by direct application of auxiliary programmed constructions (in actual fact, by the first iteration in the sense of the "indirect" version of the PIM).

In this paper we will construct an example of a DG in which (for a very obvious solution in the form of a multivalued quasi-strategy) the "direct" version of the PIM generates a solution (quasi-strategy) only after an infinite number of iterations have been performed. The situation is thus analogous, in a sense, to that considered in [14, 15]. The property in question is established using a special duality which relates "indirect" versions of the PIM, similar to those described in [8–15], and "direct" versions [18–22]; this duality was discussed in [23]. The "indirect" versions in question are objectively simpler, since they work with sets in a finite-dimensional space. Moreover, in some cases one can parametrize the iterative procedure in such a way that it reduces to a recurrence procedure on the real line (such a construction

†*Prikl. Mat. Mekh.* Vol. 66, No. 5, pp. 760–774, 2002.

was considered in [14] in a few examples, later extended to a certain class of DGs (see [24], etc.), and finally realized [25] as a computer program).

The version of the PIM considered in [18–22] is theoretical in nature and works with multivalued mappings in function spaces. In many cases, the duality of [23] essentially reduces the construction necessary to design a quasi-strategy to a form which is realizable in principle, as will be evident in the example considered below.

1. THE SIMPLEST EXAMPLE OF A CONTROL PROBLEM WITH NOISE

Let us consider the simplest control system Σ, defined by the scalar differential equation

$$\dot{x} = u + v \tag{1.1}$$

in the interval [0, 1]. The zero initial position (0, 0), i.e. $x(0) = 0$, corresponds to the main problem. For consistency with the problem considered previously [10, 14], let us assume that the admissible controls (the useful control and the noise control, respectively) are arbitrary Borel functions from [0, 1] into [–2, 2] and [–1, 1], respectively. In the first case, the set of all programmed controls is denoted by $\mathcal{U}$ (the elements of $\mathcal{U}$ are the useful Borel functions from [0, 1] into [–2, 2] and they alone). In the second case, the analogous set of possible noises $v(\cdot)$ is denoted by $\mathcal{V}$.

If the motion of system Σ (1.1) is considered over the interval $[t_*, 1]$, where $0 \leqslant t_* \leqslant 1$, from a state $x_* \in \mathbf{R}$, we shall also use as controls, functions from the set $\mathcal{U}$ and $\mathcal{V}$, respectively, although in actual fact only their restrictions to $[t_*, 1]$ are "working": if $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$, then the trajectory $x(\cdot, t_*, x_*, u(\cdot), v(\cdot)) = (x(t, t_*, x_*, u(\cdot), v(\cdot)))_{t\in[t_*,1]}$ of system Σ obviously corresponds to the function

$$t \mapsto x_* + \int_{t_*}^{t} u(\xi)d\xi + \int_{t_*}^{t} v(\xi)d\xi : [t_*, 1] \to \mathbf{R} \tag{1.2}$$

where one uses either Lebesgue integrals or, more simply, integrals in the sense of [26, p. 69] with [0, 1] equipped with the standard σ-algebra of Borel sets.

Of course, the use of such general constructions for Eq. (1.1) is in fact unnecessary; nevertheless, in that context it will be convenient to appeal to the general PIM of [10, 14]. Compared with previously considered problems [8–11, 13, 14], here we have a case in which the use of measure-controls is also superfluous, since the effect of working with the latter is achieved by working with Borel controls $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$.

In this connection, for the specific case required below, we introduce a simplified definition of the operator $\mathcal{A}_M$ of [10, 14], which is adequate for our purposes. Here M is the target set in the homing problem [3, Chap. III], defined by the condition

$$M \doteq \{(1, x) : x \in \mathbf{R},\ |x| \geqslant 1\} \tag{1.3}$$

Of course, condition (1.3) defines a subset of the position space $\mathbf{D} \doteq [0, 1] \times \mathbf{R}$. In connection with the general procedure of [10, 14] (see also [5, p. 178]), we note that in what follows we shall be considering a control problem without phase constraints, which, in the earlier notation of [10, 14], corresponds to the case $N = \mathbf{D}$ (in the more general constructions of [10, 14], the letter N denoted the set that determined the phase constraints of the corresponding control problem). In view of these circumstances, we observe that the operator $\mathcal{A}_M$, like that introduced in [10, 14], acts in the set $\mathcal{D}$ of all subsets of $\mathbf{D}$, and for $H \in \mathcal{D}$ it satisfies the condition

$$\begin{aligned}&\mathcal{A}_M(H) \doteq \{(t_*, x_*) \in H \mid \forall v(\cdot) \in \mathcal{V}\, \exists u(\cdot) \in \mathcal{U}\\ &\exists \vartheta \in [t_*, 1] : ((\vartheta, x(\vartheta, t_*, x_*, u(\cdot), v(\cdot))) \in M)\,\&\\ &\&((\xi, x(\xi, t_*, x_*, u(\cdot), v(\cdot))) \in H,\quad \forall \xi \in [t_*, \vartheta[)\}\end{aligned} \tag{1.4}$$

Let $\mathcal{F}$ be the set of all closed subsets of $\mathbf{D}$, $\mathcal{F} \subset \mathcal{D}$; for $H \in \mathcal{F}$ we have

$$\begin{aligned}&\mathcal{A}_M(H) = \{(t_*, x_*) \in H \mid \forall v(\cdot) \in \mathcal{V}\, \exists u(\cdot) \in \mathcal{U} :\\ &(|x(1, t_*, x_*, u(\cdot), v(\cdot))| \geqslant 1)\,\&\\ &\&((\xi, x(\xi, t_*, x_*, u(\cdot), v(\cdot))) \in H, \forall \xi \in [t_*, 1])\}\end{aligned} \tag{1.5}$$

It follows from (1.5) that for sets in $\mathcal{F}$ the values of the operator $\mathcal{A}_M$ defined by condition (1.4) are identical with those of the operator $\mathcal{A}$ introduced in [5, pp. 178, 179]. Here we are taking into consideration the fact that the set M, as defined by (1.3), is naturally closed in $\mathbf{D}$, as well as the fact that $\mathcal{F}$ is an invariant subspace of $\mathcal{A}_M$. Thus, in the language used in the notation of condition (1.4), the set $M = \{(1, x) : x \in \mathbf{R}, |x| \geqslant 1\}$, expressed as an iterative sequence $W_k (k = 0, 1, 2, \ldots)$, that is

$$(W_0 \doteq \mathbf{D}) \& (W_k = \mathcal{A}_M(W_{k-1}), \forall k \in \mathcal{N}) \tag{1.6}$$

where $\mathcal{N} \doteq \{1, 2 \ldots\}$, has the property of convergence to the positional absorption set W_∞: the set W_∞ is the intersection of all the sets $W_k (k = 0, 1, 2, \ldots)$.

Let us now consider a specific construction of the procedure (1.6), using the technique proposed in [14, 24]. It essentially consists of the following: all iterations in (1.6) have the same form, differing only in the value of a certain parameter. Hence the procedure (1.6) may be reduced to iterations of this parameter. We introduce the set

$$\mathcal{W} \doteq \{(t, x) \in \mathbf{D} \mid t \leqslant |x|\} \in \mathfrak{D} \tag{1.7}$$

and a mapping $\mathbf{H}$ from [0, 1] into $\mathfrak{D}$ for which

$$\mathbf{H}(\vartheta) \doteq \mathcal{W} \cup ([0, \vartheta] \times \mathbf{R}), \quad \vartheta \in [0, 1] \tag{1.8}$$

We shall consider $\mathbf{H}$ as a function of the set form in (1.6).

We will show that $W_k = \mathbf{H}(\vartheta_k)$ $(k = 0, 1, 2, \ldots)$. The corresponding sequence $(\vartheta_k)_{k=0}^{\infty}$ will be described (see also [24]). Note that by virtue of definition (1.8)

$$(W_0 = \mathbf{D} = \mathbf{H}(1)) \& (\mathcal{W} = \mathbf{H}(0)) \tag{1.9}$$

By (1.9), the construction of the sequence $(\vartheta_k)_{k=0}^{\infty}$ may be regarded as a certain passage from 1 to 0 in parameter space.

Note that for $\vartheta \in [0, 1]$ one can define the set $\mathcal{A}_M(\mathbf{H}(\vartheta))$.

We also mention one fact previously established in [14] for a slightly different (but similar) problem (see also the survey in [5, §4]); a similar proposition was presented in [24] for a more general case: if $\vartheta \in [0, 1]$, then

$$\mathcal{A}_M(\mathbf{H}(\vartheta)) = \mathbf{H}(\vartheta / 2) \tag{1.10}$$

For the sake of completeness, we will prove relation (1.10). Fix $\vartheta \in [0, 1]$. Let $(t_*, x_*) \in \mathcal{A}_M(\mathbf{H}(\vartheta))$. Then, in particular, $(t_*, x_*) \in \mathbf{H}(\vartheta)$. We will show that $(t_*, x_*) \in \mathbf{H}(\vartheta/2)$. To prove this, we will consider separately the two possibilities implied by definition (1.8): either $(t_*, x_*) \in \mathcal{W}$, in which case $(t_*, x_*) \in \mathbf{H}(\vartheta/2)$, or

$$(t_*, x_*) \in \mathbf{H}(\vartheta) \setminus \mathcal{W} \tag{1.11}$$

Only the case (1.11) requires separate consideration. Then $|x_* < t_*$. At the same time, it follows from (1.8) and (1.11) that $t_* \in [0, \upsilon]$. By the choice of (t_*, x_*), it follows from definition (1.8) that

$$\begin{gathered}\forall v(\cdot) \in \mathcal{V} \exists u(\cdot) \in \mathcal{U} : (1 \leqslant |x(1, t_*, x_*, u(\cdot), v(\cdot))|) \& \\ \&((\xi, x(\xi, t_*, x_*, u(\cdot), v(\cdot))) \in \mathbf{H}(\vartheta), \quad \forall \xi \in [t_*, 1[)\end{gathered} \tag{1.12}$$

(we recall that $|x_*| < t_* \leqslant \vartheta$). Fix $\bar{v}(\cdot) \in \mathcal{V}$. Using relation (1.12), choose $\bar{u}(\cdot) \in \mathcal{U}$ so that

$$\begin{gathered}(1 \leqslant |x(1, t_*, x_*, \bar{u}(\cdot), \bar{v}(\cdot))|) \& \\ \&((\xi, x(\xi, t_*, x_*, \bar{u}(\cdot), \bar{v}(\cdot))) \in \mathbf{H}(\vartheta), \quad \forall \xi \in [t_*, 1[)\end{gathered} \tag{1.13}$$

This implies the inequality

$$\vartheta \leqslant |x(\vartheta, t_*, x_*, \bar{u}(\cdot), \bar{v}(\cdot))| \tag{1.14}$$

In fact, for $\vartheta = 1$ inequality (1.14) is the first proposition in (1.13). Let $\vartheta < 1$. Then $\vartheta \in [t_*, 1[$ and at the same time $\forall \tau \in]\vartheta, 1[: (\tau, x(\tau, t_*, x_*, \bar{u}(\cdot), \bar{v}(\cdot))) \in \mathbf{H}(\vartheta)$.

Thus, $(\tau, x(\tau, t_*, x_*, \bar{u}(\cdot), \bar{v}(\cdot))) \in \mathcal{W}$ (see (1.8)), so that for $\tau \in]\vartheta, 1[$ we have $\tau \leqslant |x(\tau, t_*, x_*, \bar{u}(\cdot), \bar{v}(\cdot))|$.

We will assume

$$\vartheta_k \doteq \vartheta + (1-\vartheta)/k, \quad \forall k \in \mathcal{N}$$

Then $(\vartheta_k)_{k\in\mathcal{N}}$ is a sequence in $]\vartheta, 1[$ that converges to ϑ from the right. By the condition

$$\vartheta_k \leqslant |x(\vartheta_k, t_*, x_*, \bar{u}(\cdot), \bar{v}(\cdot))|, \quad \forall k \in \mathcal{N}$$

using the continuity of the trajectory, we obtain inequality (1.14) in the case $\vartheta < 1$ also.

It is useful to observe that

$$x(t_*, t_*, x_*, \bar{u}(\cdot), \bar{v}(\cdot)) = x_*, \quad |x(t_*, t_*, x_*, \bar{u}(\cdot), \bar{v}(\cdot))| < t_*$$

which, compared with inequality (1.14), yields $t_* \neq \vartheta$, that is, $t_* < \vartheta$. We recall moreover that $|x_*| < t_*$.

We introduce the notation

$$I_f(\vartheta) \doteq \int_{t_*}^{\vartheta} f(t)dt, \quad f = u, v, \bar{u}, \bar{v}, \ldots$$

It follows from inequality (1.14) that

$$\vartheta \leqslant |x_* + I_{\bar{v}}(\vartheta)| + |I_{\bar{u}}(\vartheta)| \leqslant |x_* + I_{\bar{v}}(\vartheta)| + 2(\vartheta - t_*) \tag{1.15}$$

(we recall that the function $\bar{v}(\cdot)$ was chosen arbitrarily). Thus, by inequality (1.15)

$$|x_* + I_v(\vartheta)| + 2(\vartheta - t_*) \geqslant \vartheta, \quad \forall v(\cdot) \in \mathcal{V}$$

It follows from (1.15), in particular, that

$$\forall v(\cdot) \in \mathcal{V} \exists u(\cdot) \in \mathcal{U} : |x_* + I_v(\vartheta) + I_u(\vartheta)| \geqslant \vartheta \tag{1.16}$$

(or course, this relation may also be derived from inequality (1.14)).

Let us consider the following two possibilities

$$1) \ \exists v(\cdot) \in \mathcal{V} : x_* + I_v(\vartheta) = 0,$$

$$2) \ |x_* + I_v(\vartheta)| > 0, \quad \forall v(\cdot) \in \mathcal{V}.$$

In Case 1, returning to relations (1.15) and (1.16), we obtain $t_* \leqslant \vartheta/2$.

In Case 2 we have $x_* \neq 0$. Let us choose a control $v_*(\cdot) \in \mathcal{V}$ as follows: define $v_*(t) \equiv -\operatorname{sgn} x_*$. Then

$$x_* + I_{v_*}(\vartheta) = [|x_*| - (\vartheta - t_*)] \operatorname{sgn} x_* \tag{1.17}$$

Suppose $|x_*| < \vartheta - t_*$, where, as already remarked, $\vartheta - t_* > 0$. Define a constant control by the rule

$$v_{**}(t) \equiv -x_*/(\vartheta - t_*), \quad v_{**}(\cdot) \in \mathcal{V}$$

(we already know that $|v_{**}(t)| < 1$). Then

$$x_* + I_{v_{**}}(\vartheta) = x_* - x_* = 0$$

which is impossible in Case 2. Hence $|x_*| - (\vartheta - t_*) \geqslant 0$ and, as a corollary

$$|x_* + I_{v_*}(\vartheta)| = |x_*| - (\vartheta - t_*)$$

Using the corollary to inequality (1.15), we obtain $|x_*| \geqslant t_*$, which is impossible. Thus Case 2 is impossible. We have thus shown that $t_* \leqslant v/2$ and so $(t_*, x_*) \in \mathbf{H}(\vartheta/2)$, proving that

$$\mathcal{A}_M(\mathbf{H}(\vartheta)) \subset \mathbf{H}(\vartheta/2) \tag{1.18}$$

We shall now show that in fact the right- and left-hand sides of (1.18) are equal.

Let $(t^*, x^*) \in \mathbf{H}(\vartheta/2)$. In particular, $(t^*, x^*) \in \mathbf{H}(\vartheta)$. Then, by definition (1.8), we conclude that either $(t^*, x^*) \in \mathcal{W}$ or $t^* \leqslant \vartheta/2$. Note that in the first case we define a constant control $u^* \in [-2, 2]$, putting $u^* = +2$ for $x^* \geqslant 0$ and $u^* = -2$ for $x^* < 0$. We then have

$$|x(t,t^*,x^*,u^*(\cdot),v(\cdot))|=|(x^*+u^*(t-t^*))+I_v(t)|\geq|x^*+u^*(t-t^*)|-|I_v(t)|\geq$$
$$\geq|x^*|+2(t-t^*)-(t-t^*)=|x^*|+(t-t^*)\geq t^*+(t-t^*)\geq t,\quad \forall v(\cdot)\in\mathcal{V},\ t\in[t^*,1] \tag{1.19}$$

where $u^*(\cdot)\in\mathcal{U}$ is defined as $u^*(t)\equiv u^*$, and $I_\upsilon(t)$ is defined (here and below) by integration over $[t^*, t]$. Thus, in the case $(t^*, x^*)\in\mathcal{W}$ it is true that

$$(t,x(t,t^*,x^*,u^*(\cdot),v(\cdot)))\in\mathcal{W}$$
$$\mathcal{W}\subset\mathbf{H}(\vartheta),\ \ v(\cdot)\in\mathcal{V},\ \ t\in[t^*,1]$$

In addition, it follows from relation (1.19) that

$$|x(1,t^*,x^*,u^*(\cdot),v(\cdot))|\geq 1$$

Since the choice of $\upsilon(\cdot)$ was arbitrary, consideration of condition (1.4) implies that in the case $(t^*,x^*)\in\mathcal{W}$ we have the property $(t^*,x^*)\in\mathcal{A}_M(\mathbf{H}(\vartheta))$.

It remains to consider the case when $(t^*, x^*)\notin\mathcal{W}$, that is, $|x^*|<t^*$. Then $t^*\leq\vartheta/2$. Choose $\hat{\upsilon}(\cdot)\in\mathcal{V}$ arbitrarily and, putting $y^*\doteq x^*+I_{\hat{\upsilon}}(\vartheta)$, define $\hat{u}(\cdot)\in\mathcal{U}$ by the rule: $\hat{u}(t)\doteq 2\operatorname{sgn} y^*$ for all $t\in[0, 1]$. Then we have

$$x(t,t^*,x^*,\hat{u}(\cdot),\hat{v}(\cdot))=x(\vartheta,t^*,x^*,\hat{u}(\cdot),\hat{v}(\cdot))+\int_\vartheta^t\hat{u}(\tau)d\tau+\int_\vartheta^t\hat{v}(\tau)d\tau,\ \forall t\in[\upsilon,\ 1] \tag{1.20}$$

It can be seen that for the trajectory (1.20)

$$|x(t,t^*,x^*,\hat{u}(\cdot),\ \hat{v}(\cdot))|\geq|y^*|+2(t-t^*)-(t-\vartheta)\geq t+\vartheta-2t^*\geq t,\quad \forall t\in[\vartheta,1] \tag{1.21}$$

(where we have taken into account that $t^*\leq\vartheta/2$). Thus,

$$(t,x(t,t^*,x^*,\hat{u}(\cdot),\hat{v}(\cdot)))\in\mathcal{W},\quad t\in[\vartheta,1]$$

Consequently

$$(t,x(t,t^*,x^*,\hat{u}(\cdot),\ \hat{v}(\cdot)))\in\mathbf{H}(\vartheta) \tag{1.22}$$

for all $t\in[t^*, 1]$ (see (1.8)). It follows from inequality (1.21) that

$$|x(1,t^*,x^*,\hat{u}(\cdot),\ \hat{v}(\cdot))|\geq 1 \tag{1.23}$$

It follows from relations (1.22) and (1.23) that for $(t^*,x^*)\notin\mathcal{W}$ also, $\forall\upsilon(\cdot)\in\mathcal{V}\ \exists u(\cdot)\in\mathcal{U}$

$$(|x(1,t^*,x^*,u(\cdot),v(\cdot))|\geq 1)\ \&$$
$$\&((t,x(t,t^*,x^*,u(\cdot),v(\cdot)))\in\mathbf{H}(\vartheta),\quad \forall t\in[t^*,1])$$

From (1.4) we now deduce that in all cases $(t^*,x^*)\in\mathcal{A}_M(\mathbf{H}(\vartheta))$.

Thus, $\mathbf{H}(\vartheta/2)\subset\mathcal{A}_M(\mathbf{H}(\vartheta))$. Since inclusion (1.18) has already been verified, we obtain equality (1.10).

Consider the combination of relations (1.6), (1.9) and (1.10). Then $W_0=\mathbf{H}(1)$ and

$$W_1=\mathcal{A}_M(W_0)=\mathcal{A}_M(\mathbf{H}(1))=\mathbf{H}(1/2)$$

The construction of the sequence $(W_k)_{k\in\mathcal{N}_0}$, where $\mathcal{N}_0\doteq\{0\}\cup\mathcal{N}$, now continues by induction. We know that $W_0=\mathbf{H}(1/2^0)$ and $W_1=\mathbf{H}(1/2^1)$. Let $n\in\mathcal{N}$ be such that $W_n=\mathbf{H}(1/2^n)$. It follows from equalities (1.6) and (1.10) that

$$W_{n+1}=\mathcal{A}_M(W_n)=\mathcal{A}_M(\mathbf{H}(1/2^n))=\mathbf{H}(1/2^{n+1})$$

Thus, we have shown by induction that

$$W_k=\mathbf{H}(1/2^k)=W\cup([0,1/2^k]\times\mathbf{R}),\quad \forall k\in\mathcal{N}_0 \tag{1.24}$$

In particular, $\mathcal{W}_\infty$ is the intersection of all the sets (1.24), $\mathcal{W} \subset \mathcal{W}_\infty$. But if $(t_*, x_*) \in \mathcal{W}_\infty \backslash \mathcal{W}$, then $t_* \in [0, 1/2^k]$ for all $k \in \mathcal{N}_0$, and so $t^* = 0$ and then $(t_*, x_*) = (0, x_*) \in \mathcal{W}$ by definition (1.7), which is impossible. Thus $\mathcal{W}_\infty \backslash \mathcal{W} = \emptyset$, that is, $\mathcal{W}_\infty = \mathcal{W}$ (see [14]). Consequently, we have constructed an "indirect" version (in the terminology of [19–22]) of the PIM for the specific example considered here.

2. ITERATIVE CONSTRUCTION OF A MULTIVALUED QUASI-STRATEGY

In several publications ([16, 17], etc.) use was made of a formalization of the concept of game control in the class of what are known as quasi-strategies, that is, physically realizable responses to noise control. Arbitrary responses were termed "pseudo-strategies" [17]. Use was made in [5, 8–11, 13, 14, 28, 29] of multivalued quasi-strategies (essentially from the standpoint of the constructive design of solving control procedures), which, for non-linear problems of the theory of differential games, involved extending the space of controls by admitting sliding regimes.

Iterative methods were proposed in [18–22] for constructing multivalued quasi-strategies; these methods were essentially direct versions of the PIM (in these publications, abstract control problems, not necessarily reducible to differential games, were considered; we shall not dwell on the procedures of [18–22] in all their generality). Below a more specific representation of the constructions of [19–22] will be used for the case of the problem of guidance (or homing) to M (1.3).

We take the trajectories (1.2) for $t_* = 0$ and $x_* = 0$ as basic; the "null" position $(0, 0) \in \mathcal{D}$ is chosen as the initial position. Working in accordance with the procedure of [23], we introduce a natural type of multivalued quasi-strategy (an analogue of that considered in [17]) solving the M-homing problem.

If $(t_*, x_*) \in \mathcal{D}$, $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$, then

$$x_{\leftarrow}(\cdot, t_*, x_*, u(\cdot), v(\cdot)) = (x_{\leftarrow}(t, t_*, x_*, u(\cdot), v(\cdot)))_{t \in I_0}$$

will denote the continuous function in $\mathbf{C} \doteq C([0, 1])$ for which

$$(x_{\leftarrow}(t, t_*, x_*, u(\cdot), v(\cdot)) \doteq x_*, \quad \forall t \in [0, t_*[) \,\&$$
$$\&(x_{\leftarrow}(t, t_*, x_*, u(\cdot), v(\cdot)) \doteq x(t, t_*, x_*, u(\cdot), v(\cdot)), \quad \forall t \in [t_*, 1])$$

If $t_* \in [0, 1], x_* \in \mathbf{R}$ and $v(\cdot) \in \mathcal{V}$, then [23]

$$\mathcal{S}((t_*, x_*),\ v(\cdot)) \doteq \{x_{\leftarrow}(\cdot, t_*, x_*, u(\cdot), v(\cdot)) : u(\cdot) \in \mathcal{U}\} \tag{2.1}$$

We have here introduced a pencil of trajectories continuing the trajectories (1.2) to the left, for fixed noise control $v(\cdot)$. In particular, for $t_* = 0$ and $x_* = 0$ relation (2.1) defines a pencil of ordinary trajectories. Proceeding as in [18–22], we introduce the required multivalued quasi-strategy: in terms of the family $\mathcal{P}(\mathbf{C})$ of all subsets of $\mathbf{C}$, we define an operator $\mathcal{C}$ acting from $\mathcal{V}$ into $\mathcal{P}(\mathbf{C})$ by the rule

$$\mathcal{C}(v(\cdot)) \doteq \{\mathbf{x} \in \mathcal{S}((0,0), v(\cdot)) \mid 1 \leq |\mathbf{x}(1)|\} \tag{2.2}$$

The above-mentioned multivalued version of a pseudo-strategy [17] has the form of $\mathcal{C}$.

The aim of the forthcoming constructions is to translate this "pseudo-strategy" into a multivalued quasi-strategy which, in the class of trajectories of system (1.1), will solve the problem of guaranteed realization of the condition $|x(1)| \geq 1$. In this connection we observe that

$$x_{\leftarrow}(\cdot, 0, 0, u(\cdot), v(\cdot)) = x(\cdot, 0, 0, u(\cdot), v(\cdot)), \quad u(\cdot) \in \mathcal{U}, \quad v(\cdot) \in \mathcal{V} \tag{2.3}$$

Taking this into consideration, we obtain

$$\mathcal{S}((0,0), v(\cdot)) = \{x(\cdot, 0, 0, u(\cdot), v(\cdot)) : u(\cdot) \in \mathcal{U}\}, \quad v(\cdot) \in \mathcal{V} \tag{2.4}$$

Thus, the operator $\mathcal{C}$ (2.2) may be defined (see (2.4)) in terms of trajectories of system (1.1):

$$\mathcal{C}(v(\cdot)) = \{x(\cdot, 0, 0, u(\cdot), v(\cdot)) : u(\cdot) \in \mathcal{U}, \quad 1 \leq |x(1, 0, 0, u(\cdot), v(\cdot))|\}$$

Let $\mathbf{M}(\mathcal{V}, \mathbf{C})$ be the set of all mappings from $\mathcal{V}$ into $\mathcal{P}(\mathbf{C})$. In other words, $\mathbf{M}(\mathcal{V}, \mathbf{C})$ is the set of all multivalued mappings from $\mathcal{V}$ into $\mathcal{P}(\mathbf{C})$. In particular, $\mathscr{C} \in \mathbf{M}(\mathcal{V}, \mathbf{C})$. Following the procedure of [19–22], we introduce the following special operator in $\mathbf{M}(\mathcal{V}, \mathbf{C})$, which translates $\mathscr{C}$ into a multivalued quasi-strategy guaranteed to solve the main problem. If $w \in \mathcal{V}$ and $t \in [0, 1]$, we define [22, p. 25]

$$\Omega^0(w \mid t) \doteq \{\tilde{w} \in \mathcal{V} \mid w(\tau) = \tilde{w}(\tau), \quad \forall \tau \in [0, t]\} \tag{2.5}$$

Of course, $w = v(\cdot)$ and $\tilde{w} = \tilde{v}(\cdot)$ for $\tilde{w} \in \Omega^0(w/t)$ are Borel functions from [0, 1] into [−1, 1].

Define an operator

$$\Gamma : \mathbf{M}(\mathcal{V}, \mathbf{C}) \to \mathbf{M}(\mathcal{V}, \mathbf{C}) \tag{2.6}$$

by the following rule

$$\Gamma(\alpha)(w) \doteq \{f \in \alpha(w) \mid \forall t \in [0,1] \forall \tilde{w} \in \Omega^0(w \mid t)$$

$$\exists \tilde{f} \in \alpha(\tilde{w}) : f(\tau) = \tilde{f}(\tau), \quad \forall \tau \in [0,t]\}, \quad \alpha \in \mathbf{M}(\mathcal{V}, \mathbf{C}), \quad w \in \mathcal{V} \tag{2.7}$$

With the operator (2.6), (2.7) we associate the sequence of its finite powers

$$\Gamma^k : \mathbf{M}(\mathcal{V}, \mathbf{C}) \to \mathbf{M}(\mathcal{V}, \mathbf{C}), \quad k \in \mathcal{N}_0 \tag{2.8}$$

for which, as usual

$$(\Gamma^0(\alpha) = \alpha, \quad \forall \alpha \in \mathbf{M}(\mathcal{V}, \mathbf{C})) \& (\Gamma^k = \Gamma \circ \Gamma^{k-1}, \quad \forall k \in \mathcal{N})$$

where $\circ$ denotes the operation of superposition. We complete the sequence of powers (2.8) by adding the infinite power Γ^∞ (for details, see [19–22]), defining

$$\Gamma^\infty : \mathbf{M}(\mathcal{V}, \mathbf{C}) \to \mathbf{M}(\mathcal{V}, \mathbf{C}) \tag{2.9}$$

by the rule

$$\Gamma^\infty(\alpha)(w) \doteq \bigcap_k \Gamma^k(\alpha)(w), \quad \alpha \in \mathbf{M}(\mathcal{V}, \mathbf{C}), \quad w \in \mathcal{V}$$

Here and below (in similar cases) we mean the intersection over all $k \in \mathcal{N}_0$.

In accordance with a previous result [22, Theorem 5.1], in order to achieve our goal it is important to evaluate $(\Gamma^k(\mathscr{C}))_{k \in \mathcal{N}_0}$ and the multivalued mapping $\Gamma^\infty(\mathscr{C})$. Put $\mathscr{C}_k \doteq \Gamma^k(\mathscr{C})$, $\forall k \in \mathcal{N}_0$ and, in addition, let $\mathscr{C}_\infty \doteq \Gamma^\infty(\mathscr{C})$. It is then obvious from the definition of the sequence (2.8) that

$$(\mathscr{C}_0 = \mathscr{C}) \& (\mathscr{C}_k = \Gamma(\mathscr{C}_{k-1}), \quad \forall k \in \mathcal{N}) \tag{2.10}$$

This is the basic iterative process. By definition (2.9) of Γ, we now have the following representation for $\mathscr{C}_\infty = \Gamma^\infty(\mathscr{C})$

$$\mathscr{C}_\infty(w) = \bigcap_k \mathscr{C}_k(w), \quad w \in \mathcal{V} \tag{2.11}$$

Of course, Eq. (2.11) may be used to express $\mathscr{C}_\infty$ as the limit of $(\mathscr{C}_k)_{k \in \mathcal{N}}$. To that end, we note that, by virtue of relation (2.10), $\mathscr{C}_k(v(\cdot)) \subset \mathscr{C}_{k-1}(v(\cdot))$ for $k \in \mathcal{N}$ and $v(\cdot) \in \mathcal{V}$. In combination with representation (2.11), this means [30, Chap. 1] that for $w \in \mathcal{V}$ we have monotone convergence $(\mathscr{C}_k(w))_{k \in \mathcal{N}} \downarrow \mathscr{C}_\infty(w)$. This property has been interpreted in [19–22] as pointwise convergence in the space of multivalued mappings; in the notation of [22, p. 225],

$$(\mathscr{C}_k)_{k \in \mathcal{N}} \Downarrow \mathscr{C}_\infty \tag{2.12}$$

Note that the construction of the operator Γ corresponds to that adopted in [19–22], so that the previous results may be used in the specific case under consideration.

We will now describe how the basic parameters of the general formulation [19–22] are to be specified. The set X (see [19–22]) will be the interval [0, 1]; the set Υ is defined as [−1, 1], and the family $\mathscr{X}$ of

[19–22] as $\{[0,t]: t \in [0,1]\}$; (Y, τ) of [19–22] is identified with $(\mathbf{R}, \tau_{\mathbf{R}})$, where $\tau_{\mathbf{R}}$ is the usual $|\cdot|$-topology of the real line $\mathbf{R}$; as the set Z of [19–22] we use the set $\mathbf{C} = C([0,1])$ of all continuous functions from $[0, 1]$ into $\mathbf{R}$; and the set Ω of [19–22] will be $\mathcal{V}$. Under these conditions, we have

$$\Omega_0(w \mid A) = \Omega^0(w \mid t), \quad w \in \Omega, \quad A = [0,t], \quad t \in [0,1]$$

The set $\Omega_0(w|A)$ is defined as in [21, p. 69].

Under these assumptions, with due note of definition (2.7), the operator Γ of [19–22] is given by the expressions

$$\begin{aligned} &\Gamma(\alpha)(w) = \{f \in \alpha(w) \mid \forall A \in \mathcal{X} \forall \tilde{w} \in \Omega_0(w \mid A) \\ &\exists \tilde{f} \in \alpha(\tilde{w}) : (f \mid A) = (\tilde{f} \mid A)\} = \\ &= \{f \in \alpha(w) \mid \forall t \in [0,1] \forall \tilde{w} \in \Omega^0(w \mid t) \\ &\exists \tilde{f} \in \alpha(\tilde{w}) : f(\tau) = \tilde{f}(\tau), \quad \forall \tau \in [0,t]\}, \quad \alpha \in \mathbf{M}(\mathcal{V}, \mathbf{C}), \quad w \in \mathcal{V} \end{aligned}$$

Finally, we note that the multivalued mapping $\mathcal{C}$ of (2.2) is necessarily compact-valued in the sense of the uniform convergence topology in the space $\mathbf{C}$ (in more general cases of non-linear systems one has to use generalized solutions, as was done previously in [7–11, 13, 14]).

Thus, if $v(\cdot) \in \mathcal{V}$ the set $\mathcal{C}(v(\cdot))$ is compact in $\mathbf{C}$ in the uniform convergence topology. At the same time, in previous theorems [19–22] use was made of another topology, namely, that of pointwise convergence in $Z = \mathbf{C}$ (we recall that (Y, τ) of [19–22] here is identical with $(\mathbf{R}, \tau_{\mathbf{R}})$).

It is well known, however [31], that the pointwise convergence topology in $\mathbf{C}$ is weaker than the uniform convergence topology, and then for $v(\cdot) \in \mathcal{V}$ the set $\mathcal{C}(v(\cdot))$ is also compact in the sense of the pointwise convergence topology in $\mathbf{C}$. Thus, the mapping $\mathcal{C}$ is compact-valued in $\mathbf{C}$ with the pointwise convergence topology. Thus, all conditions of Theorem 5.1 of [22] are satisfied.

We will now present the specialized version of that theorem. We introduce a (partial) order on the non-empty set $\mathbf{M}(\mathcal{V}, \mathbf{C})$, denoted by $\subseteq$, as follows:

$$\begin{aligned} &\text{def}: (\alpha_1 \subseteq \alpha_2) \Leftrightarrow (\alpha_1(v(\cdot)) \subset \alpha_2(v(\cdot)), \ \forall v(\cdot) \in \mathcal{V}) \\ &\forall \alpha_1 \in \mathbf{M}(\mathcal{V}, \mathbf{C}), \quad \forall \alpha_2 \in \mathbf{M}(\mathcal{V}, \mathbf{C}) \end{aligned} \tag{2.13}$$

In particular, as the definition of α_2 in (2.13) one can use the multivalued mapping $\mathcal{C}$.

Consider the non-empty set $\mathbf{N} \doteq \{\alpha \in \mathbf{M}(\mathcal{V}, \mathbf{C}) | \alpha = \Gamma(\alpha)\}$ (of all fixed points of Γ, that is, the set of all non-anticipatory multivalued mappings on $\Omega = \mathcal{V}$). From now on $\mathbf{N}$ will be understood only in this last-defined sense. Then (see (2.13))

$$\mathbf{N}_0[\mathcal{C}] \doteq \{\alpha \in \mathbf{N} \mid \alpha \subseteq \mathcal{C}\} \tag{2.14}$$

is the set of all non-anticipatory multi-selectors of the multivalued mapping $\mathcal{C}$. The elements of (2.14) are similar to the multivalued quasi-strategies of [5, 8–14, 28, 29], but at isolated "points" of $\mathcal{V}$ they may take the empty set $\emptyset$ as a value. In particular, the constant multivalued mapping $\alpha_\emptyset \in \mathbf{M}(\mathcal{V}, \mathbf{C})$ for which $\alpha_\emptyset(v(\cdot)) \equiv \emptyset$ is an element of (2.14). On the other hand, as shown in [21, 22], the set (2.14) has a largest element in $(\mathbf{M}(\mathcal{V}, \mathbf{C}), \subseteq)$, denoted by $(\text{na})[\mathcal{C}]$; it is the mapping from $\mathcal{V}$ into the family of all subsets of $\mathbf{C}$ for which [22, §5]

$$(\text{na})[\mathcal{C}](v(\cdot)) \doteq \bigcup_{\alpha \in \mathbf{N}_0[\mathcal{C}]} \alpha(v(\cdot)), \quad \forall v(\cdot) \in \mathcal{V} \tag{2.15}$$

It is well known [21, 22] that $(\text{na})\,[\mathcal{C}] \in \mathbf{N}_0[\mathcal{C}]$ and in addition

$$\alpha \subseteq (\text{na})[\mathcal{C}], \quad \forall \alpha \in \mathbf{N}_0[\mathcal{C}]$$

Thus, $(\text{na})[\mathcal{C}]$ is the required $\subseteq$-largest element of $\mathbf{N}_0[\mathcal{C}]$.

According to Theorem 5.1 in [22] (see also [21], Theorem 4.16), we have $\Gamma^\infty(\mathcal{C}) = (\text{na})[\mathcal{C}]$. Taking relations (2.10) and (2.11) into consideration, we have the following representation for the multivalued mapping (2.15)

$$(\mathrm{na})[\mathscr{C}] = \mathscr{C}_\infty \tag{2.16}$$

Note that representation (2.16) is necessarily a multivalued quasi-strategy, that is, for $v(\cdot) \in \mathscr{V}$, invariably $\mathscr{C}_\infty(v(\cdot)) \neq \emptyset$.

Indeed, let us consider a constant control $U_\uparrow \in \mathscr{U}$, assuming that $U_\uparrow(t) \doteq 2$ for $t \in [0, 1]$. We now introduce a mapping $\alpha_\uparrow$ from $\mathscr{V}$ into **C**, setting

$$\alpha_\uparrow(v(\cdot))(t) \doteq x(t, 0, 0, U_\uparrow, v(\cdot)) =$$

$$= \int_0^t U_\uparrow(\tau)d\tau + \int_0^t v(\tau)d\tau, \quad \forall v(\cdot) \in \mathscr{V}, \quad \forall t \in [0,1] \tag{2.17}$$

It follows from (2.17) that $\alpha_\uparrow$ is a non-anticipatory mapping in the sense of [16, 17, 27], that is, a (single-valued) quasi-strategy. Let us introduce a "fictitious" multivalued mapping $\alpha^\uparrow \in \mathbf{M}(\mathscr{V}, \mathbf{C})$ by the following rule: for $v(\cdot) \in \mathscr{V}$, the condition $\alpha^\uparrow(v(\cdot)) = \{\alpha_\uparrow(v(\cdot))\}$ is satisfied. Clearly, $\alpha^\uparrow \in \mathbf{N}$. At the same time, it follows from definition (2.17) that

$$\alpha_\uparrow(v(\cdot))(1) \geqslant \int_0^1 U_\uparrow(t)dt - 1 = 2 - 1 = 1, \quad \forall v(\cdot) \in \mathscr{V} \tag{2.18}$$

Thus, by equality (2.4)

$$\alpha_\uparrow(v(\cdot)) \in \mathscr{S}((0,0), v(\cdot)), v(\cdot) \in \mathscr{V}$$

satisfies inequality (2.18), that is (by equality (2.4)) $\alpha_\uparrow(v(\cdot)) \in \mathscr{C}(v(\cdot))$. As a corollary, $\alpha^\uparrow \subseteq \mathscr{C}$, and then, since $\alpha^\uparrow$ is non-anticipatory, it follows from definition (2.14) that $\alpha^\uparrow \in \mathbf{N}_0[\mathscr{C}]$ and so $\alpha^\uparrow \subseteq (\mathrm{na})[\mathscr{C}]$.

Hence

$$\alpha_\uparrow(v(\cdot)) \in (\mathrm{na})[\mathscr{C}](v(\cdot)), \quad \forall v(\cdot) \in \mathscr{V}$$

Thus, in representation (2.16) $C_\infty(v(\cdot)) \neq \phi$.

Since $(\mathrm{na})[\mathscr{C}]$ is a multivalued quasi-strategy, $(\mathrm{na})[\mathscr{C}]$ is a non-anticipatory multivalued mapping with non-empty values. In addition, it follows from the property $(\mathrm{na})[\mathscr{C}] \in \mathbf{N}_0[\mathscr{C}]$ that this multivalued quasi-strategy solves the problem of homing on **M** (1.3) (see the definition of $\mathscr{C}$). Finally, by definition (2.15) this quasi-strategy is largest in $(\mathbf{M}(\mathscr{V}, \mathbf{C}), \subseteq)$ among all quasi-strategies that solve the main homing problem. Relation (2.15) defines the multivalued quasi-strategy $(\mathrm{na})[\mathscr{C}]$ as the limit of an iterative procedure which is essentially a direct version of the PIM.

It is well known that in many cases (for specific differential games), procedures based on the PIM provide a solution of a DG after a finite number of iterations ([5, 8, 9], etc.). At the same time, it is sometimes necessary to construct the entire sequence of iterations [15]. However, this pertains to "indirect" versions of the PIM, that is, to slightly different iterative procedures. Examples constructed for "direct" versions of the PIM have shown that the solution may be implemented in two iterations, even counting the zeroth approximation [19, 22].

Using previously established propositions [23], we shall show that in the case considered here the "direct" version of the PIM requires (as before [15]) the construction of the entire infinite sequence of iterations. To that end, we shall introduce a more convenient representation of the operator $\mathscr{A}_M$. The following conventions will be adopted. If H is a subset of **D**, $z_* = (t_*, x_*) \in \mathbf{D}$ and $v(\cdot) \in \mathscr{V}$, then $\Pi(v(\cdot) | z_*, H)$ will denote the set of all $h \in S(z_*, v(\cdot))$ such that

$$\exists \vartheta \in [t_*, 1] : ((\vartheta, h(\vartheta)) \in M) \,\&\, ((t, h(t)) \in H, \ \forall t \in [t_*, \vartheta[) \tag{2.19}$$

Now, as in [23], let us assume that the operator $\mathbf{A}: \mathscr{P}(\mathbf{D}) \to \mathscr{P}(\mathbf{D})$ is defined by the rule

$$\mathbf{A}(H) \doteq \{z \in H \mid \Pi(v(\cdot) | z, H) \neq \emptyset, \ \forall v(\cdot) \in \mathscr{V}\}, \quad H \in \mathscr{P}(\mathbf{D}) \tag{2.20}$$

Definitions (2.19) and (2.20) correspond to the previous ones [23] provided that (in the earlier notation of [23]): $t_0 = 0$, $\vartheta_0 = 1$, $\Omega = \mathscr{V}$. By (1.4), (2.1) and (2.19), we have the obvious equality $\mathbf{A} = \mathscr{A}_M$, that is, $\mathbf{A}(H) \equiv \mathscr{A}_M(H)$.

3. CONSTRUCTION OF A MULTIVALUED QUASI-STRATEGY USING DUAL CONSTRUCTIONS OF THE PROGRAMMED ITERATION METHOD

We shall now use (1.24) and certain propositions known from [23] to verify the property

$$(\mathrm{na})[\mathscr{C}] \neq \mathscr{C}_k, \quad \forall k \in \mathbf{N}_0 \tag{3.1}$$

In this connection, we note that the mapping S defined by (2.1), which takes $\mathbf{D} \times \mathcal{V}$ into the family $\mathcal{P}'(\mathbf{C}) \doteq \mathcal{P}(\mathbf{C})\backslash\{\emptyset\}$ of all non-empty subsets of $\mathbf{C}$, satisfies all the conditions listed in [23]. Then [23, Lemma 5.1]

$$\Gamma(\Pi(\cdot \mid (0,0), H)) = \Pi(\cdot \mid (0,0), \mathbf{A}(H)), \quad H \in \mathcal{P}(\mathbf{D}) \tag{3.2}$$

By (1.3), (2.2) and (2.19), we have

$$\Pi(v(\cdot) \mid (0,0), \mathbf{D}) = \{h \in S((0,0), v(\cdot)) \,\|\, h(1) \,|\geqslant 1\} = \mathscr{C}(v(\cdot)), \quad \forall v(\cdot) \in \mathcal{V} \tag{3.3}$$

This means that $\Pi(\cdot \mid (0, 0), \mathbf{D}) = \mathscr{C}$.

It follows from expressions (3.2) and (3.3), in particular, that

$$\Gamma(\mathscr{C}) = \Gamma(\Pi(\cdot \mid (0,0), \mathbf{D})) = \Pi(\cdot \mid (0,0), \mathbf{A}(\mathbf{D}))$$

Taking relations (1.24), (2.10) and the equality $\mathbf{A} = \mathcal{A}_M$ into consideration, we obtain

$$\begin{aligned} \mathscr{C}_1 &= \Gamma(\mathscr{C}_0) = \Gamma(\mathscr{C}) = \Pi(\cdot \mid (0,0), \mathbf{A}(\mathbf{D})) = \\ &= \Pi(\cdot \mid (0,0), \mathbf{A}(W_0)) = \Pi(\cdot \mid (0,0), W_1) = \Pi(\cdot \mid (0,0), \mathbf{H}(1/2)) \end{aligned} \tag{3.4}$$

This relation may be extended by induction to any number $k \in \mathcal{N}$, in the following sense.

Proposition

$$\mathscr{C}_k = \Pi(\cdot \mid (0,0), \mathbf{H}(1/2^k)), \quad \forall k \in \mathcal{N} \tag{3.5}$$

Proof. By Corollary 5.1 and Theorem 5.1 of [23]

$$\begin{aligned} \Gamma^k(\Pi(\cdot \mid (0,0), \mathbf{D})) &= \Pi(\cdot \mid (0,0), \mathbf{A}^k(\mathbf{D})) = \\ &= \Pi(\cdot \mid (0,0), \mathcal{A}_M^k(\mathbf{D})), \quad \forall k \in \mathcal{N} \end{aligned}$$

As a corollary (see (2.10)),

$$\Gamma^k(\mathscr{C}) = \Pi(\cdot \mid (0,0), \mathcal{A}_M^k(\mathbf{D})) \tag{3.6}$$

Using relations (2.10) and (3.6), we obtain

$$\mathscr{C}_k = \Pi(\cdot \mid (0,0), \mathcal{A}_M^k(\mathbf{D}))$$

But it follows from formulae (1.6) and (1.24) that

$$\mathcal{A}_M^k(\mathbf{D}) = W_k = \mathbf{H}(1/2^k)$$

Taking relations (2.19) and (3.3) into consideration, we deduce from (3.6) that

$$\mathscr{C}_k = \Pi(\cdot \mid (0,0), \mathbf{H}(1/2^k)), \quad \forall k \in \mathcal{N}_0 \tag{3.7}$$

We now consider the representation established in [23] for the mapping $\mathscr{C}_\infty$ defined as in [22]. By Corollary 5.2 of [23] and formula (2.11), we have

$$\mathscr{C}_\infty = \Pi(\cdot \mid (0,0), \mathbf{A}^\infty(\mathbf{D})) \tag{3.8}$$

The operator $\mathbf{A}^\infty$ is defined as in [23]. Here it is essential that

$$\mathbf{A}^\infty(\mathbf{D}) = \bigcap_k \mathbf{A}^k(\mathbf{D}) = \bigcap_k \mathcal{A}^k_M(\mathbf{D}) = \bigcap_k W_k$$

Taking formula (1.24) into account, we have

$$\mathbf{A}^\infty(\mathbf{D}) = \bigcap_k \mathbf{H}(1/2^k) = \mathcal{W}_\infty = \mathcal{W} \tag{3.9}$$

But it then follows from representations (3.8) and (3.9) that

$$\mathcal{C}_\infty = \Pi(\cdot \,|\, (0,0), \mathcal{W}) \tag{3.10}$$

Since $\mathcal{W} \subset \mathbf{H}(1/2^k)$ for $k \in \mathcal{N}_0$, it follows from relations (2.19), (3.7) and (3.10) that

$$\mathcal{C}_\infty(v(\cdot)) \subset \mathcal{C}_k(v(\cdot)), \quad \forall k \in \mathcal{N}_0 \forall v(\cdot) \in \mathcal{V} \tag{3.11}$$

Theorem. The following property holds

$$\mathcal{C}_\infty \neq \mathcal{C}_k, \quad \forall k \in \mathcal{N}_0 \tag{3.12}$$

Proof. By relations (3.7) and (3.10)

$$(\mathcal{C}_k = \Pi(\cdot \,|\, (0,0), \mathbf{H}(1/2^k)), \ \forall k \in \mathcal{N}_0) \,\&\, (\mathcal{C}_\infty = \Pi(\cdot \,|\, (0,0), \mathcal{W})) \tag{3.13}$$

By virtue of inclusion (3.11), it will suffice to establish that

$$\forall k \in \mathcal{N}_0 \exists v(\cdot) \in \mathcal{V} : \mathcal{C}_k(v(\cdot)) \backslash \mathcal{C}_\infty(v(\cdot)) \neq \emptyset \tag{3.14}$$

We shall use equality (3.13). Fix $n \in \mathcal{N}_0$. Then we have

$$\mathcal{C}_n = \Pi(\cdot \,|\, (0,0), \mathbf{H}(1/2^n)), \ \mathbf{H}(1/2^n) = \\ = \mathcal{W} \cup ([0, 1/2^n] \times \mathbf{R}) \tag{3.15}$$

(the representation of the set $\mathbf{H}(1/2^n)$ follows from definition (1.8)).

Note that $w \doteq (1/2^n, 1/2^n) \in \mathcal{W}$. If $\theta \in [0, 1/2^n]$, we consider the function $f_\theta : [\theta, 1] \to [0, \infty[$ for which $f_\theta(t) \doteq 2(t - \theta)$. If $\theta \in [0, 1/2^n]$, then $1/2^n \in [\theta, 1]$, and the function value $f_\theta(1/2^n) \in [0, \infty[$ is well defined.

The simple equation $f_\theta(1/2^n) = 1/2^n$ has a solution

$$\theta_0 = 1/2^{n+1} \tag{3.16}$$

We can now establish that if $\mathbf{O} \in \mathcal{V}$ is the control defined as identically zero, then

$$\mathcal{C}_n(\mathbf{O}) \backslash \mathcal{C}_\infty(\mathbf{O}) \neq \emptyset \tag{3.17}$$

Indeed, suppose the function $u_0(\cdot) = (u_0(t) \in \mathbf{R}, 0 \leqslant t \leqslant 1) \in \mathcal{U}$ is such that

$$(u_0(t) \doteq 0, \ \forall t \in [0, \theta_0[) \,\&\, (u_0(t) \doteq 2, \ \forall t \in [\theta_0, 1]) \tag{3.18}$$

By (2.4), we conclude that the trajectory $x_0(\cdot) = (x_0(t), 0 \leqslant t \leqslant 1)$ defined by the condition $x_0(\cdot) = x(\cdot, 0, 0, u_0(\cdot), \mathbf{O})$ is an element of the set $S((0, 0), \mathbf{O})$:

$$x_0(\cdot) \in S((0,0), \mathbf{O}) \tag{3.19}$$

It follows from relation (1.2) that

$$x_0(t) = x(t, 0, 0, u_0(\cdot), \mathbf{O}) = \int_0^t u_0(t)dt, \quad \forall t \in [0,1] \tag{3.20}$$

Hence we conclude that if $t \in [0, \theta_0]$, then $x_0(t) = 0$, and for $t \in [\theta_0, 1]$

$$x_0(t) = \int_{\theta_0}^{t} 2dt = 2(t - \theta_0) = f_{\theta_0}(t) \tag{3.21}$$

It is clear that, by virtue of the second equality in (3.15), $(t, x_0(t)) \in \mathbf{H}(1/2^n)$ for $t \in [0, 1/2^n]$. Now, using relation (3.21), we have

$$x_0(t) = 2(t - \theta_0) = 2(t - 1/2^{n+1}) =$$
$$= t + (t - 1/2^n) \geqslant t, \quad t \in [1/2^n, 1]$$

This means that, by definition (1.7),

$$(t, x_0(t)) \in \mathcal{W}, \quad \forall t \in [1/2^n, 1]$$

Consequently

$$(t, x_0(t)) \in \mathbf{H}(1/2^n), \quad \forall t \in [0, 1] \tag{3.22}$$

Next, by the definition of f_{θ_0}, we have $x_0(1) = 2 - 1/2^n \geqslant 1$. Therefore, $(1, x_0(1)) \in M$. Taking relations (2.19), (3.19) and (3.22) into consideration, we have the property

$$x_0(\cdot) \in \Pi(\mathbf{O} \mid (0,0), \mathbf{H}(1/2^n))$$

that is

$$x_0(\cdot) \in \mathcal{C}_n(\mathbf{O})$$

On the other hand, $x_0(\theta_0) = 0$ for the time $\theta_0 \in]0, 1[$, defined by (3.16); hence $|x_0(\theta_0)| < \theta_0$ and consequently, by definition (1.7)

$$(\theta_0, x_0(\theta_0)) \notin \mathcal{W} \tag{3.23}$$

Returning to condition (2.19), we observe that

$$x_0(\cdot) \notin \Pi(\mathbf{O} \mid (0,0), \mathcal{W})$$

and, as a consequence, it follows from (3.13) that

$$x_0(\cdot) \notin \mathcal{C}_\infty(\mathbf{O}) \tag{3.24}$$

It follows from property (3.23), (3.24) that $x_0(\cdot) \in \mathcal{C}_n(\mathbf{O}) \setminus \mathcal{C}_\infty(\mathbf{O})$. This proves relation (3.17).

The validity of property (3.17) implies the validity of proposition (3.14), since n was chosen arbitrarily; this completes the proof of the theorem.

This research was supported financially by the Russian Foundation for Basic Research (00-01-00348, 01-01-96450) and the International Scientific-Technical Center (1293).

REFERENCES

1. ISSAACS, R., *Differential Games*. Wiley, New York, 1965.
2. KRASOVSKII, N. N., *Game Problems of the Encounter of Motions*. Nauka, Moscow, 1970.
3. KRASOVSKII, N. N. and SUBBOTIN, A. I., *Positional Differential Games*. Nauka, Moscow, 1974.
4. KRASOVSKII, N. N., *Control of a Dynamical System. The Problem of the Minimum of the Guaranteed Result*. Nauka, Moscow, 1985.
5. SUBBOTIN, A. I. and CHENTSOV, A. G., *Optimization of Guarantee in Control Problems*. Nauka, Moscow, 1981.
6. KRASOVSKII, N. N., A differential pursuit-evasion game, I. *Izv. Akad. Nauk SSSR. Tekhn. Kibernetika*, 1973, 2, 3–18.
7. KRASOVSKII, N., A differential pursuit-evasion game, II. *Izv. Akad. Nauk SSSR. Tekhn. Kibernetika*, 1973, 3, 22–42.
8. CHENTSOV, A. G., The structure of a game problem of pursuit. *Dokl. Akad. Nauk SSSR*, 1975, **224**, 6, 1272–1275.
9. CHENTSOV, A. G., A game problem of pursuit at a given instant of time. *Mat. Sbornik*, 1976, **99**, 3, 394–420.
10. CHENTSOV, A. G., A game problem of homing. *Dokl. Akad. Nauk SSSR*, 1976, **226**, 1, 73–76.
11. CHENTSOV, A. G., A game problem of homing with informational memory. *Dokl. Akad. Nauk SSSR*. 1976, **227**, 2, 306–308.
12. CHISTYAKOV, S. V., The solution of game problems of pursuit. *Prikl. Mat. Mekh.*, 1977, **41**, 5, 825–832.
13. CHENTSOV, A. G., A game problem of pursuit up to a given instant of time. *Izv. Akad. Nauk SSSR, Ser. Mat.*, 1978, **42**, 2, 455–467.

14. CHENTSOV, A. G., The method of programmed iterations for a differential game of pursuit-evasion. Sverdlovsk, 1979. Dep. at VINITI 4.06.1979, No. 1933–79.
15. CHENTSOV, A. G., An example of a non-regular differential game. *Prikl. Mat. Mekh.*, 1977, **40**, 6, 1113–1116.
16. ROXIN, E., Axiomatic approach in differential games. *J. Optimiz. Theory Appl.*, 1969, **3**, 3, 153–163.
17. ELLIOTT, R. J. and KALTON, N. J., *The Existence of Value in Differential Games* (Mem. Amer. Math. Soc., No. 126). American Mathematical Society, Providence, RI, 1972.
18. CHENTSOV, A. G., The method of programmed iterations in the class of finitely additive measure-controls. *Differents. Uravn.*, 1997, **33**, 11, 1528–1536.
19. CHENTSOV, A. G., Iterative realization of non-anticipatory multivalued mappings. *Dokl. Ross. Akad. Nuak*, 1997, **357**, 5, 595–598.
20. CHENTSOV, A. G., The question of a parallel version of an abstract analogue of the method of programmed iterations. *Dokl. Ross. Akad. Nuak*, 1998, **362**, 5, 602–605.
21. CHENTSOV, A. G., The problem of the iterative realization of non-anticipatory multivalued mappings. *Izv. Matematika*, 2000, 3, 66–76.
22. CHENTSOV, A. G., Non-anticipating selectors of set-valued mappings and iterated procedures. *Funct. Differ. Equat.*, 1999, **6**, 3–4, 249–274.
23. CHENTSOV, A. G., The problem of the compatibility of different versions of the method of programmed iterations. *Dokl. Ross. Akad. Nauk*, 2000, **372**, 5, 600–603.
24. DYATLOW, V. P. and CHENTSOV, A.. G., Control with flexible corrections with a bound on the total number of switchings. In *Gagarin Scientific Lectures on Cosmonautics and Aviation*, 1987. Nauka, Moscow, 1988, 70–75.
25. DYATLOV, V. P. and CHENTSOV, A. G., A class of linear differential games with bounded number of corrections. In *Control and Estimation in Dynamical System*. Ural'skii Nauch. Tsentr Akad. Nauk SSSR, Sverdlovsk, 1982, 9–16.
26. CHENTSOV, A. G., *Finitely Additive Measures and Relaxations of Extremal Problems*. Nauka, Ekaterinburg, 1993.
27. VARAIYA, P. and LIN, J., Existence of saddle points in differential games. *SIAM J. Control*, 1969, **7**, 1, 141–157.
28. KRASOVSKII, N. N. and CHENTSOV, A. G., On the design of differential games, I. *Probl. Control and Inform. Theory*, 1977, **6**, 5–6, 381–395.
29. KRASOVSKII. N. N. and CHENTSOV, A, G., On the design of differential games, II. *Probl. Control and Inform. Theory*, 1979, **8**, 1, 3–11.
30. NEVEU, J., *Bases Mathématiques du Calcul des Probabilités*. Masson, Paris, 1964.
31. KELLEY, J. L., *General Topology*. Van Nostrand, New York, 1957.

Translated by D.L.